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Nonlinear Kelvin and continental-shelf waves

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Equations are derived for the variation along a straight coastline of Kelvin and continental-shelf waves. It is assumed that the effects of nonlinearity and dispersion are of the same order in a small parameter ϵ defined by the equation $\epsilon^2 = f^2L^2/gH$, in which f is the Coriolis parameter, L the shelf width and H the water depth beyond the shelf. Kelvin waves are found to satisfy the Korteweg–de Vries equation, while continental-shelf waves satisfy a closely related equation. An approximate rule is derived for the variation along a real coastline of the maximum wave height for fully developed nonlinear Kelvin waves.

1. Introduction

In the original problems studied by Thomson (1879) and Robinson (1964), both the Kelvin and continental-shelf waves are non-dispersive. This means that the almost inevitable consequence of nonlinearity is shock waves and that a linear theory is of very restricted used. The more recent results of Buchwald & Adams (1968) and Larsen (1969) indicate that for some simple depth profiles there is dispersion for long waves of both classes. However, this topographic dispersion is so slight that nonlinearity could still be important. Thus it is natural to determine whether the dispersion is small for all depth profiles, and, if so, whether some form of nonlinear theory can be developed.

In §§2.1 and 3.1 it is shown that the topographic dispersion is indeed slight for all depth profiles. The shallow-water approximation, which is used here and in the four papers mentioned above, suppresses the dispersion associated with the variation of wave modes with depth. However, it can be shown that in general the ratio of the two types of dispersion is of the order L/H, where L and H are, respectively, the shelf width and the water depth beyond the continental shelf. Thus, for real coastlines the topographic dispersion is the more important, and it is justifiable to study this type of dispersion while using the shallow-water approximation.

When studying nonlinear dispersive waves, it is conventional to exploit their large wavelength when defining natural length and amplitude scales (Ursell 1953; Benjamin 1967). Here, as exemplified by the analysis of the linear equations in §§2.1 and 3.1, there is already a natural large length scale L/ϵ, where ϵ is a small parameter defined by the equation $\epsilon^2 = f^2L^2/gH$, in which f is the Coriolis parameter. Thus we are obliged to assume that the waves we are examining have length scales of this order. Likewise, if the effects of nonlinearity and dispersion are to be comparable then there are the imposed amplitude scales $\epsilon^2 H$ and $\epsilon^3 H$ for Kelvin and continental-shelf waves respectively.

The imposed amplitude scales are sufficiently large that *a priori* it would appear that nonlinearity may only be important for coastlines with narrow continental shelves. For such a shelf (of width 50 km and final depth 1 km) in mid-latitudes, the imposed amplitude scales are 2·5 metres and 0·125 metres for the two classes of waves. For those Kelvin waves associated with tides and floods a wave-height scale of 2·5 metres is very reasonable. Similarly, since the observed continental-shelf waves have been attributed to weather systems, a wave-height scale corresponding to a pressure of 12·5 millibars is acceptable (Hamon 1966).

Kelvin and continental-shelf waves have the peculiar property that they can only propagate in one direction along a coastline. Thus it is not surprising that, after all the asymptotic expansions of the following sections, it is deduced that for each class of waves the variation of wave height along the shoreline is governed by equations closely related to the equation of Korteweg & de Vries (1895), which was derived for slightly nonlinear water waves under an approximation of one-way propagation. The nonlinear term in the Korteweg–de Vries equation makes it possible for an initially small amplitude disturbance to steepen. Thus nonlinearity could be important for waves whose initial amplitudes are very small compared with the imposed wave-height scale, and consequently the nonlinear theory will have greater applicability than the initial estimates of the previous paragraph would suggest.

A simple rule which, because of the association between Kelvin waves and floods, may be of practical use is that for fully developed nonlinear Kelvin waves the maximum wave height on a coastline is proportional to the local value of

$$f^{\frac{2}{3}}\left[\int_0^\infty (H-\bar{h})\,d\bar{x}\right]^{-\frac{2}{3}},$$

where $\bar{x}$ measures distance outwards from the coast and $\bar{h}$ is the water depth at $\bar{x}$. For a real coastline the interpretation of infinite distance and H are necessarily vague, but this should not be significant. The theoretical standing of this rule is very slight since, as explained in § 2.4, there cannot be any universal rule.

2. Kelvin waves

2.1. *Linear Kelvin waves*

In non-dimensional form, linear Kelvin waves of velocity C, amplitude η and wavenumber k, propagating along a coastline of the form shown in figure 1 satisfy the shallow-water wave equations

$$\left.\begin{aligned} \frac{d}{dx}\left(h\frac{d\eta}{dx}\right)+\frac{\epsilon}{C}\frac{dh}{dx}\eta+\epsilon^2[k^2C^2-k^2h-1]\eta = 0, \\ h\left[\frac{d\eta}{dx}+\frac{\epsilon}{C}\eta\right] = 0 \quad \text{at} \quad x = 0, \\ h\frac{d\eta}{dx}-\epsilon\left[\frac{1}{C}(1-h)-\{1+k^2(1-C^2)\}^{\frac{1}{2}}\right]\eta = 0 \quad \text{at} \quad x = b_-. \end{aligned}\right\} \tag{2.1}$$

The corresponding dimensional quantities are

$$\bar{h} = Hh, \quad \bar{x} = Lx, \quad \bar{k} = \epsilon k/L, \quad \bar{C} = -C(gH)^{\frac{1}{2}}\,\mathrm{sgn}f.$$

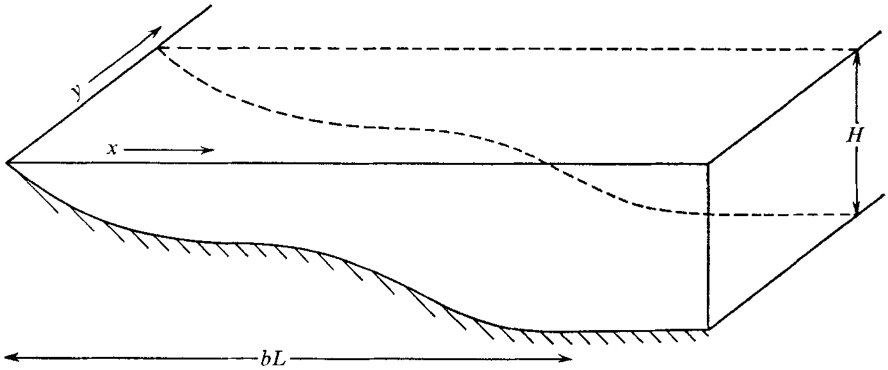

FIGURE 1. Typical depth topography.

The boundary condition at $x = b_-$ is a consequence of the requirements that the wave height and mass transport be continuous with the exponentially decaying solution which applies beyond $x = b$. Since (2.1) is a linear problem we can normalize the solution in any convenient manner; here we choose $\eta(b, \epsilon) = 1$.

In the limit as $\bar{k} \to 0$ we have a class of waves which is insensitive to the local depth variations. If these waves are to be regarded as long Kelvin waves then it is desirable that we should recover the classical Kelvin wave velocity $C = 1$. It is shown below that such a result is obtained by regarding k as fixed and formally letting $\epsilon \to 0$. However, the alternative limit procedure of regarding ϵ as fixed and letting $k \to 0$ gives different results. For this reason, and the more pragmatic reason given in §2.3, L/ϵ seems to be a natural length scale for long Kelvin waves.

For small ϵ, we deduce that η and C have expansions of the form

$$\eta = \eta_0 + \epsilon\eta_1 + \ldots, \quad C = C_0 + \epsilon C_1 + \ldots,$$

where the n_j and C_j are all independent of ϵ. The coefficients of ϵ^0 in (2.1) yield the simple problem

$$\left.\begin{aligned} \frac{d}{dx}\left(h\frac{d\eta_0}{dx}\right) &= 0, \\ h\,d\eta_0/dx = 0 \quad &\text{at} \quad x = 0, \\ h\,d\eta_0/dx = 0 \quad &\text{at} \quad x = b_-, \end{aligned}\right\}$$

which has the solution $\eta_0 = 1$.

The coefficients of ϵ in (2.1) yield the problem

$$\left.\begin{aligned} \frac{d}{dx}\left(h\frac{d\eta_1}{dx}\right) &= -\frac{1}{C_0}\frac{dh}{dx}, \\ h\,d\eta_1/dx = -h/C_0 \quad &\text{at} \quad x = 0, \\ h\,d\eta_1/dx = [(1/C_0)(1-h) - \{1+k^2(1-C_0^2)\}^{\frac{1}{2}}] \quad &\text{at} \quad x = b_-. \end{aligned}\right\}$$

These equations can only have a solution if the inhomogeneous terms are orthogonal to the zero-order solution η_0 (i.e. if we integrate the differential equation from 0 to b and use the boundary conditions we find a relationship between the inhomogeneous terms). This constraint reduces to

$$C_0\{1 + k^2(1 - C_0^2)\}^{\frac{1}{2}} = 1,$$

which has two roots:
$$C_0 = 1, \quad C_0|k| = 1.$$

We take the first root as the second corresponds to a wave of inertial frequency. The solution to the first-order equations is then

$$\eta_1 = b - x.$$

From the next two orders in ϵ, the orthogonality constraint yields the results

$$C_1 = -\int_0^b (1-h)\,dx = -\alpha \quad \text{(say)}$$

and
$$C_2 = -\tfrac{1}{2}\alpha^2 k^2 + \int_0^b \left[x(1-h) + \alpha h - \int_0^x (1-h)\,dx' \right] dx,$$

where, as above, the possibility of a wave of inertial frequency has been rejected.

Since C_1 is independent of k the first-order correction to the wave velocity applies equally to all wavenumbers and is not associated with dispersion. In contrast, unless α is zero, C_2 does depend upon k and there is dispersion at second order in ϵ. As a consequence of this delayed appearance of dispersion, the following calculations for nonlinear Kelvin waves have to be carried through a large number of steps before a balance between the effects of nonlinearity and dispersion becomes possible. Of course this also reduces the necessary scale of the nonlinearity and makes nonlinear effects correspondingly important along real coastlines.

2.2. *Equations of motion*

The results concerning linear Kelvin waves enable us to select an effective means of studying nonlinear Kelvin waves. For example, we shall measure the shape of waves from stretched axes whose origin is moving with the non-dispersive velocity $1-\epsilon\alpha$, i.e.
$$Y = ((1-\epsilon\alpha)t + \epsilon y \operatorname{sgn} f),$$

and we measure the development due to nonlinearity and dispersion on the slow time scale
$$T = \epsilon^2 t,$$

where y and t are non-dimensionalized with respect to L and $|f|$ respectively. Two major changes in the calculations from the linear case are that explicit solutions need to be found in the outer region beyond the continental shelf and that we shall use the matching principle for asymptotic expansions instead of boundary conditions at a particular line (Van Dyke 1964).

If the effects of nonlinearity and dispersion are of the same order in ϵ, then the vertically averaged momentum and continuity equations for the inner region take the form
$$\left.\begin{aligned} \eta_x - \epsilon v + \epsilon^2 u_Y - \epsilon^3 \alpha u_Y + \epsilon^4[u_T + uu_x + vu_Y] &= 0,\\ \eta_Y + v_Y + \epsilon(u - \alpha v_Y) + \epsilon^2[v_T + uv_x + vv_Y] &= 0,\\ (hu)_x + (hv)_Y + \eta_Y - \epsilon\alpha\eta_Y + \epsilon^2[\eta_T + (\eta u)_x + (\eta v)_Y] &= 0,\\ hu = 0 \quad \text{at} \quad x &= 0. \end{aligned}\right\} \tag{2.2}$$

The dimensional wave height and horizontal velocities are

$$\bar{\eta} = \epsilon^2 H\eta, \quad \bar{u} = \epsilon^3 u(gH)^{\frac{1}{2}} \operatorname{sgn} f, \quad \bar{v} = \epsilon^2 v(gH)^{\frac{1}{2}} \operatorname{sgn} f.$$

For the outer region, we replace the inner co-ordinate x by the stretched co-ordinate

$$X = \epsilon x.$$

The non-dimensional outer equations then take the form

$$\left.\begin{aligned} \zeta_X - V + \epsilon U_Y - \epsilon^2\alpha U_Y + \epsilon^3[U_T + VU_Y] + \epsilon^4 UU_X = 0, \\ \zeta_Y + V_Y + \epsilon(U - \alpha V_Y) + \epsilon^2[V_T + VV_Y] + \epsilon^3 UV_X = 0, \\ V_Y + \zeta_Y + \epsilon(U_X - \alpha\zeta_Y) + \epsilon^2[\zeta_T + (\zeta V)_Y] + \epsilon^3(\zeta U)_X = 0, \\ \zeta, U, V \to 0 \quad \text{as} \quad X \to \infty, \end{aligned}\right\} \tag{2.3}$$

where the symbols ζ, U, V have been used to distinguish the outer variables from their inner counterparts η, u, v respectively.

It is clear that for small ϵ the dependent variables once again have expansions of the form

$$\eta = \eta_0 + \epsilon\eta_1 + \ldots,$$

where the η_j are all independent of ϵ.

2.3. *Matched asymptotic expansions*

The coefficients of ϵ^0 in (2.2) yield the leading inner problem

$$\eta_{0x} = 0, \quad \eta_{0Y} + v_{0Y} = 0, \quad (hu_0)_x + (hv_0)_Y + \eta_{0Y} = 0, \quad hu_0 = 0 \quad \text{at} \quad x = 0,$$

which has the simple solution

$$\eta_0 = A_0, \quad u_0 = -A_{0Y}\frac{1}{h}\int_0^x (1-h)\,dx', \quad v_0 = -A_0, \tag{2.4}$$

where A_0 is an undetermined function of Y and T, and a steady current contribution to v_0 has been suppressed on the hypothesis that the waves are of finite extent.

The coefficients of ϵ^0 in (2.3) yield the leading outer equations

$$\zeta_{0X} - V_0 = 0, \quad \zeta_{0Y} + V_{0Y} = 0, \quad V_{0Y} + \zeta_{0Y} = 0, \quad \zeta_0, V_0 \to 0 \quad \text{as} \quad X \to \infty,$$

whose solution is

$$\zeta_0 = B_0 e^{-X}, \quad V_0 = -B_0 e^{-X}, \tag{2.5}$$

where B_0 is an undetermined function of Y and T. At this stage it is extremely simple to apply the asymptotic matching principle to the one-term inner and one-term outer solutions (2.4) and (2.5), and we deduce that

$$B_0 = A_0.$$

The coefficients of ϵ in (2.2) yield the problem

$$\left.\begin{aligned} \eta_{1x} = -A_0, \\ \eta_{1Y} + v_{1Y} = A_{0Y}\left[\frac{1}{h}\int_0^x (1-h)\,dx' - \alpha\right], \\ (hu_1)_x + (hv_1) + \eta_{1Y} = \alpha A_{0Y}, \\ hu_1 = 0 \quad \text{at} \quad x = 0. \end{aligned}\right\}$$

The solutions are

$$\eta_1 = A_1 - xA_0,$$

$$u_1 = -A_{1Y}\frac{1}{h}\int_0^x (1-h)\,dx' + A_{0Y}\left\{\frac{\alpha x}{h} + \frac{1}{h}\int_0^x \left[x'(1-h) + \alpha h - \int_0^{x'} (1-h)\,dx''\right]dx'\right\},$$

$$v_1 = -A_1 + A_0\left\{x - \alpha + \frac{1}{h}\int_0^x (1-h)\,dx'\right\},$$

where A_1 is an undetermined function. In anticipation of the next matching, we now use (2.4) to show that the two-term outer expansion of the two-term inner solutions for η and v are

$$\eta \sim A_0(1-X) + \epsilon A_1, \quad v \sim -A_0(1-X) - \epsilon A_1.$$

The coefficients of ϵ in (2.3) yield the equations

$$\left.\begin{aligned} \zeta_{1X} - V_1 &= -U_{0Y}, \\ \zeta_{1T} + V_{1Y} &= -U_0 - \alpha A_{0Y}\, e^{-X}, \\ V_{1T} + \zeta_{1Y} &= -U_{0X} + \alpha A_{0T}\, e^{-X}, \\ \zeta_1, V_1, U_0 &\to 0 \quad \text{as} \quad X \to \infty. \end{aligned}\right\}$$

We observe that the second and third of the above equations are consistent with each other only if

$$U_{0X} - U_0 = 2\alpha A_{0Y}\, e^{-X},$$

which, together with the conditions as $X \to \infty$, implies that

$$U_0 = -\alpha A_{0Y}\, e^{-X}.$$

It now follows that the solutions for ζ_1 and V_1 are

$$\left.\begin{aligned} \zeta_1 &= B_1 e^{-X} + A_{0YY}\, Xe^{-X}, \\ V_1 &= -B_1 e^{-X} - A_{0YY}\, Xe^{-X}, \end{aligned}\right\} \tag{2.6}$$

where B_1 is an undetermined function of Y and T.

From (2.5) and (2.6), we find that the two-term inner expansion of the two-term outer solutions for ζ and V are

$$\zeta \sim A_0(1-X) + \epsilon B_1, \quad V \sim -A_0(1-X) - \epsilon B_1,$$

where, for covenience, the expansions have been re-expressed in outer variables. Applying the asymptotic matching principle to the two-term inner and outer solutions, we deduce that

$$B_1 = A_1.$$

At the next order in ϵ, the equations and solutions become exceedingly lengthy because of the new terms involving nonlinearity and dispersion. However, there is nothing different in principle from the previous equations, so we shall proceed directly to the matching procedure. The three-term outer expansions of the three-term inner solutions for η and v_T are

$$\eta \sim A_0(1 - X + \tfrac{1}{2}X^2) + \epsilon\{A_1(1-X) + \alpha A_{0YY}X\} + \epsilon^2\{A_2 + \beta A_0 + \beta A_{0YY}\},$$

$$v_T \sim -A_{0T}(1 - X + \tfrac{1}{2}X^2) - \epsilon\{A_{1T}(1-X) + \alpha A_{0TYY}X\}$$
$$-\epsilon^2\{A_{2T} + (\beta+\gamma)A_{0T} + \beta A_{0TYY} - A_{0T} + \tfrac{1}{2}(A_0^2)_Y\},$$

where β and γ are constants defined by

$$\beta = \int_0^\infty \left[\frac{1}{h}\int_0^x (1-h)\,dx' - \alpha\right] dx,$$

$$\gamma = \int_0^\infty \left[x(1-h) + \alpha h - \int_0^x (1-h)\,dx'\right] dx.$$

The three-term inner expansions of the three-term outer solutions for ζ and V_Y are

$$\zeta \sim A_0(1-X-\tfrac{1}{2}X) + \epsilon\{A_1(1-X) + \alpha A_{0YY}X\} + \epsilon^2\{B_2 - \tfrac{1}{2}(A_0^2)_{YY}\},$$

$$V_Y \sim -A_{0Y}(1-X+\tfrac{1}{2}X^2) - \epsilon\{A_{1Y}(1-X) + \alpha A_{0YYY}X\}$$
$$-\epsilon^2\{B_{2Y} - \tfrac{1}{2}(A_0^2)_{YY} - \tfrac{1}{2}\alpha^2 A_{0YYY}\},$$

where, for convenience, the expansions have been re-expressed in outer variables. Applying the asymptotic matching principle to the three-term inner and outer solutions, we deduce that

$$B_2 = A_2 + \beta A_0 + \beta A_{0YYY} + \tfrac{1}{2}(A_0^2)_{YY}$$

and
$$-A_{0T} + \gamma A_{0Y} + \tfrac{1}{2}(A_0^2)_Y + \tfrac{1}{2}\alpha^2 A_{0YYY} = 0. \qquad (2.7)$$

The crucial point in the above analysis (which enables us to factor out the X variation) is that, although the three leading outer equations are not independent, they only involve the two variables ζ_0 and V_0. Had we taken the length scale of nonlinear Kelvin waves to be $\epsilon^{-\frac{1}{2}}L$, which would make the velocity of linear waves equal to $1-\epsilon(\alpha+\frac{1}{2}\alpha^2k^2)+\ldots$, then the three outer equations would not be independent, but would involve the three variables ζ_0, U_0 and V_0. The complete nonlinear analysis would then lead to the formidable problem

$$\left.\begin{aligned} &2\zeta_{0TY} - \zeta_{0XX} + \zeta_0 - \tfrac{3}{2}\zeta_{0YY}^2 = 0,\\ &\zeta_0 \to 0, \quad \text{as} \quad X \to \infty,\\ &\zeta_{0X} + \zeta_0 - \alpha\zeta_{0YY} \to 0 \quad \text{as} \quad X \to 0,\end{aligned}\right\} \qquad (2.8)$$

where ζ is related to the dimensional wave height by the scaling ϵH. The complexity of (2.8) together with the large imposed wave height scale make this alternative theory most uninteresting.

2.4. *Wave development*

The evolution of nonlinear Kelvin waves is described by (2.7), which is a simple variant of the equation of Korteweg & de Vries (1895). Qualitatively, we can deduce that a smooth wave profile will steepen owing to the 'advection' term $(\frac{1}{2}A_0^2)_Y$ whereas a jagged wave profile will become smoother owing to the 'dispersion' term $\frac{1}{2}A_{0YYY}$. From numerical solutions and as yet incomplete theoretical work (Zabusky 1967; Gardner *et al.* 1967), it seems that a balance is quite rapidly achieved between advection and dispersion, and a sequence of solitary waves is formed. For (2.7), the solitary wave solutions are described by the equation

$$A_0 = 6\alpha^2\kappa^2\,\mathrm{sech}^2\,(\kappa Y - \kappa[\gamma + 2\alpha^2\kappa^2]T),$$

where κ is an arbitrary constant.

For a single solitary wave propagating along a straight coastline there is a simple relationship between the maximum wave height at the shoreline and the total wave energy. If we assume that along a real coastline a single solitary wave remains a single solitary wave then from the conservation of energy we obtain

$$\eta_{\max} \propto g^{-1}(\text{total energy})^{\frac{2}{3}} f^{\frac{2}{3}} = \left[\int_0^\infty (H-\bar{h})\, d\bar{x}\right]^{-\frac{2}{3}}.$$

Unfortunately this simple derivation is not valid, since the number and sizes of the distinct solitary waves will vary along the coastline as the parameters change in value. A more rigorous analysis would merely demonstrate the absence of any general rule since, according to a result of Gardner *et al.* (1967), the size of the largest solitary wave which develops along a straight section of coastline equals the largest eigenvalue of a Schrödinger equation involving α as a scaling parameter and the initial wave shape as the potential; an additional complication is that the initial wave shape will necessarily depend on the properties of previous sections of coastline.

Although the Korteweg–de Vries equation may accurately describe the development of long waves, the non-physical properties which it attributes to short waves can have undesirable consequences (Benjamin, Bona & Mahony 1972). In contrast, an equation of Peregrine (1966) has ideal properties for both numerical and analytic studies yet describes the development of long waves to the same order of accuracy as the Korteweg–de Vries equation. In the moving stretched co-ordinates the appropriate modified version of (2.7) is

$$-A_{0T} + \gamma A_{0Y} + \tfrac{1}{2}(A_0^2)_Y + \tfrac{1}{2}\alpha^2\left[(1-\epsilon\alpha)\frac{\partial}{\partial Y} + \epsilon^2\frac{\partial}{\partial T}\right]A_{0YY} = 0.$$

Munk, Snodgrass & Wimbush (1970) have shown that between one and two thirds of the tidal amplitude on the California coastline can be attributed to Kelvin waves. Thus it is important to know when a linear theory can accurately describe such periodic Kelvin waves. For waves of frequency ω the ratio of non-linear to dispersive terms in (2.7) is given by

$$\bar{\eta} : \omega^2\alpha^2 L^2/g.$$

For semi-diurnal tides along a shelf of width 50 km this ratio is

$$\bar{\eta} : \alpha^2\,\mathrm{m},$$

and consequently a linear description of typical tides could be very inaccurate.

3. Continental-shelf waves

3.1. *Linear continental-shelf waves*

In non-dimensional form, linear continental-shelf waves of velocity c and wave-number k_1 propagating along a coastline of the form shown in figure 1 satisfy the shallow-water wave equations

$$\left.\begin{aligned} \frac{d}{dx}\left(\eta\frac{d\eta}{dx}\right) + \frac{1}{c}\frac{dh}{dx}\eta + \epsilon^2[-k^2h - 1 + \epsilon^2k^2c^2]\eta &= 0, \\ h\left[\frac{d\eta}{dx} + \frac{1}{c}\eta\right] = 0 \quad \text{at} \quad x &= 0, \\ h\frac{d\eta}{dx} - \frac{1}{c}(1-h)\eta + \epsilon\{1+k^2-\epsilon^2k^2c^2\}^{\frac{1}{2}}\eta = 0 \quad \text{at} \quad x &= b_-. \end{aligned}\right\} \qquad (3.1)$$

Since this is a linear problem, we can normalize the solutions in any convenient manner; here we choose

$$\int_0^b h\left(\frac{d\eta}{dx}\right)^2 dx = 1.$$

The crucial difference between these equations and the corresponding equations for Kelvin waves is that the dimensional velocity is given by $\bar{c} = -\epsilon c(gH)^{\frac{1}{2}}\,\mathrm{sgn}\,f$.

For small ϵ we deduce that η and c have expansions of the form

$$\eta = \eta_0 + \epsilon\eta_1 + \ldots, \quad c = c_0 + \epsilon c_1 + \ldots,$$

where the η_j and c_j are all independent of ϵ. The coefficients of ϵ^0 in (3.1) yield the Sturm–Liouville eigenvalue problem

$$\left.\begin{aligned} \frac{d}{dx}\left(h\frac{d\eta_0}{dx}\right) + \frac{1}{c_0}\frac{dh}{dx}\eta_0 = 0,& \\ h\left[\frac{d\eta_0}{dx} + \frac{1}{c_0}\eta_0\right] = 0 \quad \text{at} \quad x = 0,& \\ h\frac{d\eta_0}{dx} - \frac{1}{c_0}(1-h)\eta_0 = 0 \quad \text{at} \quad x = b_-.& \end{aligned}\right\}$$

Thus η_0 will equal one of the normalized eigenfunctions $\phi(x)$, and c_0 the corresponding eigenvalue. For most real depth profiles c_0 will be positive in view of the identity

$$c_0 = \int_0^b \frac{dh}{dx}\phi^2\,dx + [(1-h)\phi^2]_{x=b_-} + [h\phi^2]_{x=0}.$$

The coefficients of ϵ in (3.1) yield the problem

$$\frac{d}{dx}\left(h\frac{d\eta_1}{dx}\right) + \frac{1}{c_0}\frac{dh}{dx}\eta_1 = \frac{c_1}{c_0^2}\frac{dh}{dx}\phi,$$

$$h\left[\frac{d\eta_1}{dx} + \frac{1}{c_0}\eta_1\right] = \frac{c_1}{c_0^2}h\phi \quad \text{at} \quad x = 0,$$

$$h\frac{d\eta_1}{dx} - \frac{1}{c_0}(1-h)\eta_1 = -\frac{c_1}{c_0^2}(1-h)\phi - \{1+k^2\}^{\frac{1}{2}}\phi \quad \text{at} \quad x = b_-.$$

These equations can only have a solution if the inhomogeneous terms are orthogonal to the zero-order solution ϕ. This constraint reduces to

$$c_1 = -c_0\{1+k^2\}^{\frac{1}{2}}\phi(b)^2,$$

which shows that for any realistic depth profile there is dispersion at the first order in ϵ. Also, this expression for c_1 indicates that L/ϵ is indeed a natural length scale for long continental-shelf waves, since other scalings would in effect approximate $\{1+k^2\}^{\frac{1}{2}}$ by $1+\frac{1}{2}k^2$ or $|k|$, with a consequent loss of information.

It should be noted that the assumed scalings in the above perturbation analysis are equivalent to the two physical assumptions (*a*) that the horizontal divergence is small and (*b*) that the wavelength is large. Assumption (*a*) has been discussed by Longuet-Higgins (1968) and (*b*) concentrates attention upon one particular part of the complete spectrum described by Buchwald & Adams

(1968) for a particular geometry. The waves produced by the perturbation analysis are consistent with Hamon's (1966) observations but not with those of Cartwright (1969).

3.2. *Equations of motion*

There are three features which make the analysis of nonlinear continental-shelf waves considerably easier than the corresponding analysis of nonlinear Kelvin waves. First, it is profitable to work with a single equation involving the wave height as the only scalar dependent variable. Second, we shall not use matched asymptotic expansions since the solutions are extremely insensitive to the line at which matching is performed and, as explained by Benjamin (1967) in a mathematically similar problem, the final equations governing the development of nonlinear dispersive waves can be extended in an obvious manner if the depth profiles are only asymptotically flat. Third, dispersion is involved at the first order in ϵ.

If the effects of nonlinearity and dispersion are of the same order in ϵ, then the vertically averaged momentum and continuity equations for the outer region can be combined into the scalar equation

$$\left.\begin{aligned} \left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}-1\right)\left(\zeta+\frac{\epsilon}{c_0}\zeta_T\right)+O(\epsilon^2) = 0,\\ \zeta = \eta \quad \text{at} \quad X = 0,\\ \zeta \to 0 \quad \text{as} \quad X \to \infty. \end{aligned}\right\} \tag{3.2}$$

Here, X, Y and T are stretched co-ordinates defined by the equations

$$X = \epsilon(x-b), \quad Y = \epsilon(c_0 t + y\,\mathrm{sgn} f), \quad T = \epsilon^2 t.$$

The corresponding inner equations are

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left(h\frac{\partial}{\partial x}\eta_Y\right)+\frac{1}{c_0}h_x\eta_Y+\frac{\epsilon}{c_0}\frac{\partial}{\partial x}(hN(\eta))+O(\epsilon^2) = 0,\\ h\left[\frac{\partial}{\partial x}\eta_Y+\frac{1}{c_0}\eta_Y+\frac{\epsilon}{c_0}N(\eta)+O(\epsilon^2)\right] = 0 \quad \text{at} \quad x = 0,\\ h\frac{\partial}{\partial x}\eta_Y-\frac{1}{c_0}(1-h)\eta_Y+\epsilon\left(\frac{h}{c_0}N(\eta)-\zeta_{XY}\right)+O(\epsilon^2) = 0 \quad \text{at} \quad x = b_-, \end{aligned}\right\} \tag{3.3}$$

where $N(\eta)$ denotes the combination of terms

$$\eta_{Tx}+\eta\eta_{xY}-\eta_Y\eta_{xx}-c_0\eta_{xY}\eta_{xx}.$$

In these equations the dimensional wave height is assumed to be given by

$$\overline{\eta} = \epsilon^3\eta H.$$

From (3.2) and (3.3) it is clear that for small ϵ the dependent variable has an expansion of the form

$$\eta = \eta_0+\epsilon\eta_1+\ldots,$$

where the η_j are independent of ϵ.

3.3. *Inner and outer solutions*

The coefficients of ϵ^0 in (3.3) yield the equations

$$\frac{\partial}{\partial x}\left(h\frac{\partial}{\partial x}\eta_{0Y}\right)+\frac{1}{c_0}h_x\eta_{0Y} = 0,$$

$$h\left[\frac{\partial}{\partial x}\eta_{0Y}+\frac{1}{c_0}\eta_{0Y}\right] = 0 \quad \text{at} \quad x = 0,$$

$$h\frac{\partial}{\partial x}\eta_{0Y}-\frac{1}{c_0}(1-h)\eta_{0Y} = 0 \quad \text{at} \quad x = b_-,$$

from which it follows that η_0 is some multiple $A_0(Y,T)$ of the eigenfunction $\phi(x)$ corresponding to the eigenvalue c_0.

The coefficients of ϵ^0 in (3.2) yield the problem

$$\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}-1\right)\zeta_0 = 0,$$

$$\zeta_0 = A_0\phi(b) \quad \text{at} \quad X = 0,$$

$$\zeta \to 0 \quad \text{as} \quad X \to \infty,$$

from which it follows that the Fourier transforms $\hat{\zeta}_0(k)$ and $\hat{A}_0(k)$ of ζ_0 and A_0 are related by the equation

$$\hat{\zeta}_0 = \exp\left(-\{1+k^2\}^{\frac{1}{2}}X\right)\hat{A}_0\phi(b).$$

The coefficients of ϵ^1 in (3.3) yield the inhomogeneous equations

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left(h\frac{\partial}{\partial x}\eta_{1Y}\right)+\frac{1}{c_0}h_x\eta_{1Y} &= -\frac{1}{c_0}\frac{\partial}{\partial x}(hN(A_0\phi)),\\ h\left[\frac{\partial}{\partial x}\eta_{1Y}+\frac{1}{c_0}\eta_{1Y}\right] &= -\frac{h}{c_0}N(A_0\phi) \quad \text{at} \quad x = 0,\\ h\frac{\partial}{\partial x}\eta_{1Y}-\frac{1}{c_0}(1-h)\eta_{1Y} &= -\frac{h}{c_0}N(A_0\phi)+\phi(b)\,M(A_{0Y}) \quad \text{at} \quad x = b_-,\end{aligned}\right\}$$

where M is the pseudo-differential operator with symbol $\{1+k^2\}^{\frac{1}{2}}$, i.e.

$$\hat{M}(B) = \{1+k^2\}^{\frac{1}{2}}\hat{B}.$$

These equations can only have a solution if the inhomogeneous terms are orthogonal to the eigenfunction $\phi(x)$. This constraint reduces to

$$A_{0T}+\delta A_0 A_{0Y}-c_0\phi(b)^2\partial M(A_0)/\partial Y = 0, \tag{3.4}$$

where δ is a constant defined by the integral

$$\int_0^b h[\phi_x^3-\phi\phi_x\phi_{xx}-c_0\phi_x^2\phi_{xx}]\,dx.$$

For depth profiles which are only asymptotically flat, (3.4) is still applicable provided that c_0, ϕ and δ are defined for the interval $(0,\infty)$ instead of the interval $(0,b)$.

3.4. *Wave development*

There is very little that can be said directly concerning the solutions of (3.4). However, for wave profiles which are dominated by small wavenumber components, the symbol of the operator M (that is, its Fourier transform $\{1+k^2\}^{\frac{1}{2}}$) can be approximated by $1+\frac{1}{2}k^2$; (3.4) is thereby approximated by the Korteweg–de Vries equation. Similarly, for waves profiles which are dominated by large wavenumber components, the symbol of M can be approximated by $|k|$ and (3.4) is approximated by the Benjamin–Davis equation (Benjamin 1967; Davis & Acrivos 1967). Both of the limiting equations have solitary wave solutions so it is natural to presume that (3.4) will have a solitary wave solution. Unfortunately I have been unable to derive a description of these solitary waves.

Following Peregrine (1966) and Benjamin, Bona & Mahony (1972), we note that for both numerical and analytic studies it is desirable to replace (3.4) by the modified equation

$$A_{0T}+\delta A_0 A_{0Y}-\phi^2(b)\left[c_0\frac{\partial}{\partial Y}+\epsilon\frac{\partial}{\partial T}\right]MA_0=0.$$

4. Discussion

For both Kelvin and continental-shelf waves the balance between non-linear and dispersive effects takes place on a time scale of order $\epsilon^{-2}f^{-1}$. During this time the waves will have moved a distance of order $\epsilon^{-3}L$ and $\epsilon^{-2}L$, respectively, for the two classes of waves. These distances are so great that we are forced to conclude that a nonlinear theory is only necessary if the coastline is a closed curve. For example, tides propagating in a sea which has a narrow shelf and is almost closed can be expected to have wave profiles which differ significantly from the profiles that would be predicted by a linear theory. Thus, although this paper may have demonstrated that a theory for nonlinear Kelvin and continental-shelf waves is both possible and desirable, the detailed results in this paper are not well suited for comparison with any real situation, owing to the neglect of curvature, changes in depth profile and changes in the Coriolis parameter.

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